HW 18 Sol

1) What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?

**Ans:** There are 1 \(\leq 1 \times 3, 2 \times 3, \ldots, 33 \times 3 \leq 100\) hence there are 33 numbers divisible by 3. 

Hence Prob = \(\frac{33}{100}\)

2) What is the probability of these events when we randomly select a permutation of \(\{1, 2, \ldots, n\}\) where \(n \geq 4\)?

a) 1 precedes 2.
**Ans:** By symmetry \(\frac{1}{2}\)

b) 2 precedes 1.
**Ans:** By symmetry \(\frac{1}{2}\)

c) 1 immediately precedes 2.

**Ans:** Remember permutation means each number appears in each location same number of times. So let us pretend that 1 and 2 are glued (try some examples). There are now \((n-1)\) numbers, hence \((n-1)!\) Permutations in which 1 and 2 are together.

\[
\text{Prob} = \frac{(n-1)!}{n!} = \frac{1}{n}
\]

d) \(n\) precedes 1 and \(n-1\) precedes 2.

**Ans:**
Observe \(\frac{1}{2}\) permutations \(n\) precedes 1. (by symmetry)
Observe from the above permutations \(\frac{1}{2}\) of them: \((n-1)\) precedes 2 (by symmetry)

Hence : \(\frac{1}{4}\)

e) \(n\) precedes 1 and \(n\) precedes 2.

**Ans:** Observe by symmetry:

\((n\ \text{precedes}\ \{1, 2\})\) \(\cdot\) \((1\ \text{precedes}\ \{n, 2\})\) \(\cdot\) \((2\ \text{precedes}\ \{1, n\})\) should be equal

Hence prob = \(1/3\)
3) Suppose $E$ and $F$ are events such that $P(E) = 0.8$ and $P(F) = 0.6$.
Show that:
$P(E \cup F) \geq 0.8$ and $P(E \cap F) \geq 0.4$

Ans: $P(E \cup F) \geq P(E) = 0.8$

Note: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
$\Rightarrow P(E \cup F) = 0.8 + 0.6 - P(E \cap F) \leq 1$ (probability has to be less than 1)
$\Rightarrow P(E \cap F) \geq 0.4$

4) Urn 1 contains 2 blue tokens and 8 red tokens; urn 2 contains 12 blue tokens and 3 red tokens.
You roll a die to determine which urn to choose: if you roll a 1 or 2 you choose urn 1; if you roll a 3, 4, 5, or 6 you choose urn 2. Once the urn is chosen, you draw out a token at random from that urn. Given that the token is blue, what is the probability that the token came from urn 1?

Ans:

Let $E = \{ \text{Chosen ball is blue} \}$
$F = \{ \text{came from urn 1} \}$

$P(F/E) = \frac{P(E/F) \cdot P(F)}{P(E/F) \cdot P(F) + P(E/F^c) \cdot P(F^c)}$
$P(E/F) = \frac{2}{10} = \frac{1}{5}$; $P(F) = \frac{2}{6} = \frac{1}{3}$; $P(F^c) = \frac{2}{3}$; $P(E/F^c) = \frac{12}{15} = \frac{4}{5}$

Hence $P(F/E) = \frac{1/5 \cdot 1/3}{1/5 \cdot 1/3 + 4/5 \cdot 2/3} = \frac{1}{9}$