NOTATION. \[ N = \{0, 1, 2, \ldots \} \]
\[ \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \]
\[ \mathbb{R} = \{ \text{all the real numbers} \} \]

DEF. RATIONAL NUMBERS

A rational number is a number \( \frac{a}{b} \) such that \( a \in \mathbb{N} \) and \( b \in \mathbb{N} \setminus \{0\} \) and \( \gcd(a, b) = 1 \).

\[ \text{Eg: } \frac{a}{b} = \frac{1}{2} \to \frac{2}{4} , \frac{5}{9} , \frac{12}{8} \]

NOTATION. \( \mathbb{Q} = \{ \text{rational numbers} \} \)

DEF. IRATIONAL \( \equiv \) NOT RATIONAL.

Theorem. \( \sqrt{2} \) is irrational.

Proof. Assume \( \sqrt{2} \) is rational.

\[ \therefore \sqrt{2} = \frac{a}{b} \quad (a, b \in \mathbb{N}) \quad (b \neq 0) \]
\[ \gcd(a, b) = 1 \]

\[ \sqrt{2} \cdot \frac{a}{b} = \frac{a^2}{b^2} \to \frac{2 \cdot b^2}{a^2} = \frac{a^2}{b^2} \]

\[ \text{hence } a^2 \text{ is even } \Rightarrow a \text{ is even} \]

Then \( a = 2k_1 \quad (k_1 \in \mathbb{N}) \)

\[ 2 \cdot b^2 = (2k_1)^2 = 4k_1^2 \]
\[ 2b^2 = (2k_1)^2 = 4k_1^2 \]

\[ b^2 = 2k_1^2 \]

\[ \Rightarrow b^2 \text{ is even} \Rightarrow b \text{ is even} \]

\[ b = 2k_2 \quad (k_2 \in \mathbb{N}) \]

\[ \text{hence } \gcd(9, 5) \geq 2 \]

Contradiction to the assumption: \( s_2 \) is rational.

Here \( s_2 \) irrational

\[ s_2 \implies \text{rational} \]

\[ \neg p \implies \neg q \]

\[ s_2 \land \neg q \]

\[ (p \land q) \implies F \]

\[ p \implies q \]

\[ \neg (p \land q) \lor F \]

\[ \equiv \neg (p \land q) \]

\[ \neg (p \lor q) \]

\[ p \lor q \equiv p \implies q \]

RESULT: There exists an irrational number such that \((x, y)\) is rational.

Solution: Let \( x = s_2 \), \( y = s_2 \).
Consider \( x^2 = 12 \) valid.

Consider \( (\sqrt{12})^2 = 12 \) invalid.

Chap 2

**Def.** Set \( S \) is a collection of objects.

\( S = \{1, 2, \ldots \} \) is the natural numbers.

**Def.** Union, intersection, complement, difference.

**Def.** Union

Let \( A, B \) be two sets. Then

\[ A \cup B = \{ x \mid (x \in A) \lor (x \in B) \} \]

\( S \) if

\[ A = \{1, 3, 5, 6\} \quad B = \{1, 5, 7, 9, 10, 2\} \]

\[ A \cup B = \{1, 2, 3, 5, 6, 7, 9, 10, 2\} \]

**Inters:** \( A \cap B = \{ x \mid (x \in A) \land (x \in B) \} \)

\( A \cap B = \{1, 5, 6\} \)
Diff: $A - 0 = \{ x \mid x \in A \land x \neq 0 \}$
$A - 0 = \{ 1, 3, 5, 6 \}$
$0 - A = \{ 7, 9, 10, 2 \}$

\[ \overline{A} = A^c = \begin{cases} 
1, 3, 5, 6 \\
A 
\end{cases} \]

\[ \text{univ} = \mathbb{N} \]

Complement:

\[ \overline{A} = A^c = \begin{cases} 
1, 3, 5, 6 \\
A 
\end{cases} \]

\[ = \{ 0, 2, 4, 7, 8, 9, \ldots \} \]

\[ X \]

DEF: $A$, $B$ be two sets. We say $A$ is a subset of $B$: $(A \subseteq B)$

\[ A \subseteq B \quad \iff \quad \forall x \in A \implies x \in B \]

Strict subset $A \subset B \iff (A \subseteq B) \land (A \neq B)$

\[ (P \cup Q) = P \cup Q \quad \iff \quad (A \cup B)^c = A^c \cap B^c \]

\[ (P \cap Q) = P \cap Q \quad \iff \quad (A \cap B)^c = A^c \cup B^c \]

Proof: $(A \cup B)^c = A^c \cap B^c$ 

\[ \text{must show:} \quad (A \cup B)^c \subseteq (A^c \cap B^c) \quad \text{--- 1} \]

\[ (A^c \cap B^c) \subseteq (A \cup B)^c \quad \text{--- 2} \]

Let $x \in (A \cup B)^c \implies x \notin (A \cup B)$
\[ A = \{ a, b, c \} \]

\[ \emptyset \subseteq A \iff \{ x \in \emptyset \Rightarrow \exists a \in A \} \]

\[ \mathcal{B} = \{ \emptyset, 001, 010, 011, 100, 101, 110, 111 \} \]

**Cartesian Product**

\[ A \times \mathcal{B} = \{ (a, b) \mid (a \in A) \land (b \in \mathcal{B}) \} \]

**Example**

\[ A = \{ 1, 2, 3 \}, \quad \mathcal{B} = \{ 1, 2, 3 \} \]

\[ A \times \mathcal{B} = \{ (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3) \} \]
**DEF.** A is a set. Then

\[ |A| \] the size is \# of elements in the set if the set is finite.

**Ex:** \[ A = \{1, 2, 3\} \] \[ |A| = 3 \]

**DEF.** Function

Let \( A \subseteq B \) be two sets. Then \( f \) is a function from \( A \) to \( B \) (denoted \( f : A \rightarrow B \)) if and only if:

1. \((\forall x \in A) \ f(x) \in B\)
2. \((\forall x \in A) \ (\exists y \in B)\) such that \( f(x) = y \land y \) is unique.

**Ex.** Let \( f : N \rightarrow N \) \( \text{N} \)}
Let \( f : \mathbb{N} \rightarrow \mathbb{N} \)

\[ f(n) = 2n \]

**DEF:** A function \( f : A \rightarrow B \) is said to be one to one if and only if

\[ (\forall x_1, x_2 \in A) \quad [x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)] \]

\[ f(x_1, x_2) = \underbrace{\ldots}_{2 	imes 2} \]