*10. Show that in any set of \( n + 1 \) positive integers not exceeding \( 2n \) there must be two that are relatively prime.

Ans: In class

Form the \( n \) boxes
\[(1,2) (3,4) (5,6), \ldots (2n-1,2n)\]

Now since you have \( n+1 \) numbers \( \leq 2n \)

by PHP there must be 2 numbers

in some box. Observe \((a, a+1)\) are

relatively prime. \(\times\)

42. A witness to a hit-and-run accident tells the police that the license plate of the car in the accident, which contains three letters followed by three digits, starts with the letters AS and contains both the digits 1 and 2. How many different license plates can fit this description?

Ans: In class

\[\text{Case (1) assume distinct from } 1, 2 \text{ (3,4,5,6,7,8,9,0)}\]
\[\left[ \begin{array}{c} 1 \times 2 \times 2 \times 2 \\ 1, 2, 1, 2, 1, 2 \end{array} \right]\]
\[\text{Case (2) The digit is not } 1, 2, \text{ or (1,2,2)}\]
\[\text{case: } \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3}{3! \times 2!} = 2 \]

\[\text{case: } \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3}{3! \times 2!} = 2 \]
CHAPTER 7 Discrete Probability

We start with finite probability:

Definition: Suppose we perform an experiment. Then collect all the possible outcomes (finite), this set is called a sample space.

Definition: If $S$ is a finite nonempty sample space of equally likely outcomes, and $E$ is an event, that is, a subset of $S$, then the probability of $E$ is:

$$p(E) = \frac{|E|}{|S|}.$$ 

Observe: $0 \leq p(E) \leq 1$

Example 3:
In a lottery, players win a large prize when they pick four digits that match, in the correct order, four digits selected by a random mechanical process. A smaller prize is won if only three digits are matched. What is the probability that a player wins the large prize? What is the probability that a player wins the small prize?

Ans: Large Prize: $S = \{ 10^4 \}$

$E = \{ \text{only one way} \} \quad P(E) = \frac{1}{10^4} = 0.0001$

Small prize?

$E = \{ (1,2,4, x), (1,2, x, 4), (1, x, 3, 4), (x, 2, 3, 4) \}$

Exact three correct, $x$ is exactly wrong (9 choices)

Total $9 \times 4 = 36$

$P(E) = \frac{36}{10^4} = 0.0036$

Example 4:
There are many lotteries now that award enormous prizes to people who correctly choose a set of six numbers out of the first $n$ positive integers, where $n$ is usually between 30 and 60. What is the probability that a person picks the correct six numbers out of 40?

Ans:
Solution: There is only one winning combination. The total number of ways to choose six numbers out of 40 is

$$C(40, 6) = 40!/[34! \cdot 6!] = 3,838,380.$$
#2. What is the probability that a fair die comes up six when it is rolled?

**Ans:** Sample space \( S = \{1, 2, 3, 4, 5, 6\} \)

Event \( E = \{6\} \) (\( E \subseteq S \))

\[
p(E) = \frac{|E|}{|S|} = \frac{1}{6}
\]

#6. What is the probability that a card selected at random from a standard deck of 52 cards is an ace or a heart?

**Ans:** \( S = \{1, 2, \ldots, 52\} \Rightarrow |S| = 52 \)

\( E = \{4 \text{ Aces (Which includes Ace of hear) } + (13-1)\} \)

\( |E| = 16 \)

\[
P( E ) = \frac{16}{52} = 0.3077
\]

**Question:** What is the probability that when two dice are thrown the sum comes up 7?

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**Ans:** \( P ( E ) = \frac{6}{36} = 0.1667 \)

**Question:** What is the probability that when two dice are thrown the sum at least 8?
P ( E ) = 15 / 36 = 0.4167

**Probabilities of Complements and Unions of Events**

**THEOREM 1:** Let E be an event in a sample space S. The probability of the event \( \bar{E} = S - E \), the complementary event of E, is given by:
\[ p(\bar{E}) = 1 - p(E). \]

**Proof:**
To find the probability of the event \( \bar{E} = S - E \), note that \( |\bar{E}| = |S| - |E| \). Hence,
\[ p(\bar{E}) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E). \]

**EXAMPLE 8:** A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

Ans: Size sample space \( 2^{10} = 1024 \)

Probability of no zero bits (i.e. 1111111111) = \( \frac{1}{1024} \)

Probability of at least one bit 0 = 1 - Probability if no zero bits

= 1 - \( \frac{1}{1024} \) = 0.999

**THEOREM 2:** Let \( E_1 \) and \( E_2 \) be events in the sample space S. Then
\[ p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2). \]

**Proof:**
\[ p(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{(|E_1| + |E_2| - |E_1 \cap E_2|)}{|S|} = \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} = p(E_1) + p(E_2) - p(E_1 \cap E_2). \]

**EXAMPLE 9:** What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

Ans: Let \( E_1 \) be the event that the integer selected at random is divisible by 2,
\( E_2 \) be the event that it is divisible by 5.

\( E_1 \cup E_2 \) is the event that it is divisible by either 2 or 5. \( E_1 \cap E_2 \) is the event that it is divisible by both 2 and 5, or equivalently, that it is divisible by 10.

\(|E_1| = 50, |E_2| = 20, \text{ and } |E_1 \cap E_2| = 10\)

\[ E_1 \cup E_2 = \]
\[ p(E_1 \cup E_2) = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{100} = \frac{50 + 20 - 10}{100} = \frac{60}{100} = 0.6 \]
**Probabilistic Reasoning**

A common problem is determining which of two events is more likely. Analyzing the probabilities of such events can be tricky. Example 10 describes a problem of this type. It discusses a famous problem originating with the television game show Let’s Make a Deal and named after the host of the show, Monty Hall.

Behind one of these doors is a car.