6.5 Generalized Permutations and Combinations

**EXAMPLE 1:** How many strings of length \( r \) can be formed from the uppercase letters of the English alphabet?

**Solution:** By the product rule, because there are 26 uppercase English letters, and because each letter can be used repeatedly, we see that there are \( 26^r \) strings of uppercase English letters of length \( r \).

**THEOREM 1:** The number of \( r \)-permutations of a set of \( n \) objects with repetition allowed is \( n^r \).

**Proof:** There are \( n \) ways to select an element of the set for each of the \( r \) positions in the \( r \)-permutation when repetition is allowed, because for each choice all \( n \) objects are available. Hence, by the product rule there are \( n^r \) \( r \)-permutations when repetition is allowed.

**Combinations with Repetition**

Consider these examples of combinations with repetition of elements allowed.

**EXAMPLE 2:** How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

**Solution:** To solve this problem we list all the ways possible to select the fruit. There are 15 ways:

- 4 apples
- 3 apples, 1 orange
- 3 oranges, 1 pear
- 2 apples, 2 oranges
- 2 apples, 1 orange, 1 pear
- 4 oranges
- 3 apples, 1 pear
- 3 pears, 1 apple
- 2 apples, 2 peares
- 2 oranges, 1 apple, 1 pear
- 4 pears
- 3 oranges, 1 apple
- 3 pears, 1 orange
- 2 oranges, 2 pears
- 2 pears, 1 apple, 1 orange

The solution is the number of 4-combinations with repetition allowed from a three-element set, \{apple, orange, pear\}.

Now let's see how to do this mathematically:
There are three categories of fruits (n elements).  
From the picture you need 2 bars to separate them (n-1 bars).

So to choose four fruits (r fruits), we can think of placing 4 fruits.  
Two bars indicating our selection:

EX:  
* * * *

This selects:  1 apple  2 oranges  1 pear

Ex:  
* / * * *

This selects:  0 apple  4 oranges  0 pear

Ex:  
/ * * * *

This selects:  3 apple  1 oranges  0 pear

Ex:  
/ * * * *

This selects:  0 apple  0 oranges  4 pears

Hence we can think of this process as having 6 positions (r + n-1) And choosing 2 places (n-1). That is

C(6,2)  \{ C(r + n-1, n-1) \}

C(6,2) = 6!/(4!*2!) = 15

In general  C(r + n -1, r) = C( r+n-1, n-1) (they are both the same).
THEOREM 2
There are \( C(n + r - 1, r) = C(n + r - 1, n - 1) \) \( r \)-combinations from a set with \( n \) elements when repetition of elements is allowed.

**Proof:** Each \( r \)-combination of a set with \( n \) elements when repetition is allowed can be represented by a list of \( n - 1 \) bars and \( r \) stars. The \( n - 1 \) bars are used to mark off \( n \) different cells, with the \( i \)th cell containing a star for each time the \( i \)th element of the set occurs in the combination. For instance, a \( 6 \)-combination of a set with four elements is represented with three bars and six stars. Here

\[
** | * | * | * *
\]
represents the combination containing exactly two of the first element, one of the second element, none of the third element, and three of the fourth element of the set.

As we have seen, each different list containing \( n - 1 \) bars and \( r \) stars corresponds to an \( r \)-combination of the set with \( n \) elements, when repetition is allowed. The number of such lists is \( C(n - 1 + r, r) \), because each list corresponds to a choice of the \( r \) positions to place the \( r \) stars from the \( n - 1 + r \) positions that contain \( r \) stars and \( n - 1 \) bars. The number of such lists is also equal to \( C(n - 1 + r, n - 1) \), because each list corresponds to a choice of the \( n - 1 \) positions to place the \( n - 1 \) bars.

EXAMPLE 4
Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

**Solution:** The number of ways to choose six cookies is the number of \( 6 \)-combinations of a set with four elements. From Theorem 2 this equals \( C(4 + 6 - 1, 6) = C(9, 6) \). Because

\[
C(9, 6) = C(9, 3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84,
\]
there are 84 different ways to choose the six cookies.

EXAMPLE 5
How many solutions does the equation

\[
x_1 + x_2 + x_3 = 11
\]

have, where \( x_1, x_2, \) and \( x_3 \) are nonnegative integers?

**Solution:** To count the number of solutions, we note that a solution corresponds to a way of selecting 11 items from a set with three elements so that \( x_1 \) items of type one, \( x_2 \) items of type two, and \( x_3 \) items of type three are chosen. Hence, the number of solutions is equal to the number of \( 11 \)-combinations with repetition allowed from a set with three elements. From Theorem 2 it follows that there are

\[
C(3 + 11 - 1, 11) = C(13, 11) = C(13, 2) = \frac{13 \cdot 12}{1 \cdot 2} = 78
\]

? Now suppose we want \( x_1 \geq 1; x_2 \geq 2; x_3 \geq 3 \); How many solutions?

**Sol:** Since we have \( x_1, x_2, \) and \( x_3 \) to have at least \((1+2+3)\) values; we have only \((11-6) = 5\) values to play with;

\[
C(5 + 3 -1, 5) = 21
\]
#8. How many different ways are there to choose a dozen donuts from the 21 varieties at a donut shop?

\[ r = 12; \quad n = 21; \]
C(12 + 21 -1, 12 ) = C(32,12 ) = 31*29*4*26*5*23*21 = 225,792,840

#10. A croissant shop has plain croissants, cherry croissants, chocolate croissants, almond croissants, apple croissants, and broccoli croissants. How many ways are there to choose

**Given:** n = 6;

a) a dozen croissants?
   **Ans:** r = 12;  \( C(12+6-1, 5) = C(17,5) = 17!/(12!*5!) = 6,188 \)

b) three dozen croissants?
   **Ans:** r = 36;  \( C(36+6-1,5) \)

c) two dozen croissants with at least two of each kind?
   **Ans:** Start with \( r = 24 \); but you have to choose 2 of each kind.
   So 12 is already selected and you are left with \( (24-12) \) to select  
   \( r = 24-12 = 12 \)  \( C(12+6-1, 5) = 6188 \) (part 1)

d) two dozen croissants with no more than two broccoli croissants?
   **Ans:** Start with \( r = 24 \); and \( n = 6 \);
   You are allowed to pick:
   Zero broccoli:  \( r =24; \ n = 5 \)  \( C(24+5-1, 4) \)
   One broccoli:  \( r = 23; \ n = 5 \)  \( C(23+5-1, 4) \)
   Two broccoli:  \( r = 22; \ n = 5 \)  \( C(22+5-1, 4) \)

   Total = \( C(28,4) + C(27,4) + C(26,4) \); Note once you pick broccoli you have only 5 type of croissants left.

Another Way: ( #of ways to Pick 2 dozen broccoli) - (#ways to pick 3 or more broccoli)  
\( C(24+6-1, 5) - (21+6-1, 5) \)

Yo can compute an see if you get the same answer!!!!

e) two dozen croissants with at least five chocolate croissants and at least three almond croissants?
   **Ans:**  \( r = 24 - 5 - 3=16 \)  \( => \)  \( C(16+6-1,5) \)

f) two dozen croissants with at least one plain croissant, at least two cherry croissants, at least three chocolate croissants, at least one almond croissant, at least two apple croissants, and no more than three broccoli
croissants?

Ans: \{\text{choose 1Plain + 2Cherry +3Choc +1 Alm + 2app + no restrictions on broccoli}\}

subtract
\{\text{choose 1Plain + 2Cherry +3Choc +1 Alm + 2app + 4 broccoli}\}

\{r = 24 - 1 -2 -3 -1-2 = 15 , n = 6 \} - \{ r = 15-4=11 , n = 6 \}
C(15+6-1,5) - C(11+6-1, 5) = C(20,5) - C(16,5) = 20!/(5!*15!) - 16!/(5!*11!) = 11,136

16. How many solutions are there to the equation

\[ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29, \]

where \( x_i, i = 1, 2, 3, 4, 5, 6, \) is a nonnegative integer such that

a) \( x_i > 1 \) for \( i = 1, 2, 3, 4, 5, 6? \)
b) \( x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, |x_4 \geq 4, x_5 > 5, \) and \( x_6 \geq 6? \)
c) \( x_1 \leq 5? \)
d) \( x_1 < 8 \) and \( x_2 > 8? \)

Ans: \( n = 6 \)

(a) Start with \( r = 29; \) since each \( x_i \) requires 2; you use up 12 so \( r = 29-12=17 \)

\[ C(17+6-1, 5) = 22!/(5!*17!) = 26,334 \]

(b) \( r = 29 - 1 - 2 - 3 - 4- 6- 6 = 7 \) \( \implies \) \( C(7+6-1, 5) = 12!/(5!*7!) = 792 \)

(c) \{all possible solutions with out restrictions\} - \{ possible solutions with \( x_1 \geq 6 \}\)
\{r = 29, n = 6 \} - \{ r = 29-6 = 23 , n = 6 \}
\[ C(29+ 6-1, 5) - C(23+ 6-1, 5) = C(34,5) - C(28,5) = 34!/(5!*29!) - 28!/(5!*23!) = 179,976 \]

(d) \{solutions with \( x_2 \geq 9 \} - \{ \text{solutions with } x_2 \geq 9 \text{ and } x_1 \geq 8 \}
\{r = 29-9 = 20 , n = 6 \} - \{r = 29 - 9 -8 = 12 ; n = 6 \}
\[ C(20+6-1,5) - C(12+6-1, 5) = C(25, 5) - C(17,5) = 25!/(5!*20!) - 17!/(5!*12!) = 46,942 \]

#25. How many positive integers less than 1,000,000 have the sum of their digits equal to 19?

Ans: since its less than equal to 999999 we can represent each
digit by \( x_1, x_2, x_3, x_4, x_5, \) \( x_6; \)
(Eg: \( 234523 \implies x_1 = 2; x_2 = 3; x_3 = 4; x_4 = 5; x_5 = 2 ; x_6 = 3 \)
and \( x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 19 )\)

lets compute all possible numbers:
\[ C(19+6-1, 5) = C(24,5) = 24!/(5!*19!) = 42,504 \]
Note these will include values for \( x_1 \geq 10; x_2 \geq 10,... x_6 \geq 10; \)
We will subtract these values. Let's compute how many possibilities with $x_1 \geq 10$;

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 9 \quad \implies \quad C(9+6-1, 5) = C(14, 5) = \frac{14!}{5!9!} = 2,002$$

Notes soon as $x_1 \geq 10$ the rest of $x_2, x_3, ..x_6$ cannot be $\geq 10$;

So the total is $2002 \times 6 = 12012$

Hence $42504 - 12012 = 30492$

**Permutations with Indistinguishable Objects**

EXAMPLE 7 How many different strings can be made by reordering the letters of the word SUCCESS?

Solution: Because some of the letters of SUCCESS are the same, the answer is not given by the number of permutations of seven letters. This word contains three S's, two C's, one U, and one E. To determine the number of different strings that can be made by reordering the letters, first note that the three S's can be placed among the seven positions in $C(7, 3)$ different ways, leaving four positions free. Then the two Cs can be placed in $C(4, 2)$ ways, leaving two free positions. The U can be placed in $C(2, 1)$ ways, leaving just one position free. Hence E can be placed in $C(1, 1)$ way. Consequently, from the product rule, the number of different strings that can be made is

$$C(7, 3)C(4, 2)C(2, 1)C(1, 1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!}$$

$$= \frac{7!}{3!2!1!1!}$$

$$= 420.$$
The number of different permutations of \( n \) objects, where there are \( n_1 \) indistinguishable objects of type 1, \( n_2 \) indistinguishable objects of type 2, \ldots, and \( n_k \) indistinguishable objects of type \( k \), is

\[
\frac{n!}{n_1!n_2! \cdots n_k!}.
\]

**Proof:** To determine the number of permutations, first note that the \( n_1 \) objects of type one can be placed among the \( n \) positions in \( C(n, n_1) \) ways, leaving \( n - n_1 \) positions free. Then the objects of type two can be placed in \( C(n - n_1, n_2) \) ways, leaving \( n - n_1 - n_2 \) positions free. Continue placing the objects of type three, \ldots, type \( k - 1 \), until at the last stage, \( n_k \) objects of type \( k \) can be placed in \( C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k) \) ways. Hence, by the product rule, the total number of different permutations is

\[
C(n, n_1)C(n - n_1, n_2) \cdots C(n - n_1 - \cdots - n_{k-1}, n_k)
\]

\[
= \frac{n!}{n_1!(n - n_1)!} \frac{(n - n_1)!}{n_2!(n - n_1 - n_2)!} \cdots \frac{(n - n_1 - \cdots - n_{k-1})!}{n_k!0!}
\]

\[
= \frac{n!}{n_1!n_2! \cdots n_k!}.
\]

So this one for example you assume this is first 3 my second 5 this at four that is that's one way same as and how many

Page 433:

**#30.** How many different strings can be made from the letters in MISSISSIPPI, using all the letters?

**Ans:** \( n = 11 \) (11 letters)

\( n_1 = 4 \) (four s); \( n_2 = 4 \) (four i); \( n_3 = 2 \) (2 p)

\[
11! / (4! * 4! * 2!) = 34,650
\]

**#32.** How many different strings can be made from the letters in AARDVARK, using all the letters, if all three As must be consecutive?

**Ans:** Treat 3 A's as one letter;

\( 6! / 2! = 360 \)

**#40.** How many ways are there to travel in x y z w space from the origin \((0, 0, 0)\) to the point \((4, 3, 5, 4)\) by taking steps one unit in the positive x, positive y, positive z, or positive w direction?

**Ans:** One way is \( x x x x y y y z z z z w w w \) (4 x's, 3 y's, 5 z's and 4 w's)

How many ways to rearrange this? \( 16! / (4! * 3! * 5! * 4!) = 50,450,400 \)