March 16

6.1 The Basics of Counting

THE PRODUCT RULE: Suppose that a procedure can be broken down into a sequence of two tasks. If there are \( n_1 \) ways to do the first task and for each of these ways of doing the first task, there are \( n_2 \) ways to do the second task, then there are \( n_1 \times n_2 \) ways to do the procedure.

EXAMPLE 1
A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

\textit{Solution:} The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are \( 12 \times 11 = 132 \) ways to assign offices to these two employees.

EXAMPLE 2
The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

\textit{Solution:} The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are \( 26 \times 100 = 2600 \) different ways that a chair can be labeled. Therefore, the largest number of chairs that can be labeled differently is 2600.

EXAMPLE 3
There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

\textit{Solution:} The procedure of choosing a port consists of two tasks, first picking a microcomputer and then picking a port on this microcomputer. Because there are 32 ways to choose the microcomputer and 24 ways to choose the port no matter which microcomputer has been selected, the product rule shows that there are \( 32 \times 24 = 768 \) ports.

Page 396 Exercises.

\#2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

\textbf{Ans:} By the product rule \( 27 \times 37 \)

\#3. A multiple-choice test contains 10 questions. There are four possible answers for each question.
a) In how many ways can a student answer the questions on the test if the student answers every question?

Ans: Each question has 4 possibilities; hence 10 questions will have
\[ 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^{10} \]

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

Ans: \( 5^{10} \)

#4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

Ans: \( 12 \times 2 \times 3 \)

#6. There are four major auto routes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?

Ans: \( 4 \times 6 \)

#10. How many bit strings are there of length eight?

Ans: Each bit can take 2 value \{0, 1\}. Hence \( 2^8 \)

#14 How many bit strings of length \( n \), where \( n \) is a positive integer, start and end with 1s?

Ans: Since tow its are fixed; you are left with \((n-2)\) bits to assign values. hence \( 2^{(n-2)} \)

**Example 6**  **Counting Functions**  How many functions are there from a set with \( m \) elements to a set with \( n \) elements?

**Solution:** A function corresponds to a choice of one of the \( n \) elements in the codomain for each of the \( m \) elements in the domain. Hence, by the product rule there are \( n \cdot n \cdot \cdots \cdot n = n^m \) functions from a set with \( m \) elements to one with \( n \) elements. For example, there are \( 5^3 = 125 \) different functions from a set with three elements to a set with five elements.

**Example 7**  **Counting One-to-One Functions**  How many one-to-one functions are there from a set with \( m \) elements to one with \( n \) elements?
**Solution:** First note that when \( m > n \) there are no one-to-one functions from a set with \( m \) elements to a set with \( n \) elements.

Now let \( m \leq n \). Suppose the elements in the domain are \( a_1, a_2, \ldots, a_m \). There are \( n \) ways to choose the value of the function at \( a_1 \). Because the function is one-to-one, the value of the function at \( a_2 \) can be picked in \( n - 1 \) ways (because the value used for \( a_1 \) cannot be used again). In general, the value of the function at \( a_k \) can be chosen in \( n - k + 1 \) ways. By the product rule, there are \( n(n - 1)(n - 2) \cdots (n - m + 1) \) one-to-one functions from a set with \( m \) elements to one with \( n \) elements.

For example, there are \( 5 \cdot 4 \cdot 3 = 60 \) one-to-one functions from a set with three elements to a set with five elements.

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**THE SUM RULE** If a task can be done either in one of \( n_1 \) ways or in one of \( n_2 \) ways, where none of the set of \( n_1 \) ways is the same as any of the set of \( n_2 \) ways, then there are \( n_1 + n_2 \) ways to do the task.

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**EXAMPLE 12** Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

**Solution:** There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule \( \star \) follows that there are \( 37 + 83 = 120 \) possible ways to pick this representative.

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**EXAMPLE 16** Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

**Solution:** Let \( P \) be the total number of possible passwords, and let \( P_6, P_7, \) and \( P_8 \) denote the number of possible passwords of length 6, 7, and 8, respectively. By the sum rule, \( P = P_6 + P_7 + P_8 \). We will now find \( P_6, P_7, \) and \( P_8 \). Finding \( P_6 \) directly is difficult. To find \( P_6 \) it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits. By the product rule, the number of strings of six characters is \( 36^6 \), and the number of strings with no digits is \( 26^6 \). Hence,

\[
P_6 = 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560.
\]

Similarly, we have

\[
P_7 = 36^7 - 26^7 = 78,364,164,096 - 8,031,810,176 = 70,332,353,920
\]

and

\[
P_8 = 36^8 - 26^8 = 2,821,109,907,456 - 208,827,064,576 = 2,612,282,842,880.
\]

Consequently,

\[
P = P_6 + P_7 + P_8 = 2,684,483,063,360.
\]
THE SUBTRACTION RULE If a task can be done in either $n_1$ ways or $n_2$ ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

The subtraction rule is also known as the principle of inclusion–exclusion, especially when it is used to count the number of elements in the union of two sets. Suppose that $A_1$ and $A_2$ are sets. Then, there are $|A_1|$ ways to select an element from $A_1$ and $|A_2|$ ways to select an element from $A_2$. The number of ways to select an element from $A_1$ or from $A_2$, that is, the number of ways to select an element from their union, is the sum of the number of ways to select an element from $A_1$ and the number of ways to select an element from $A_2$, minus the number of ways to select an element that is in both $A_1$ and $A_2$. Because there are $|A_1 \cup A_2|$ ways to select an element in either $A_1$ or in $A_2$, and $|A_1 \cap A_2|$ ways to select an element common to both sets, we have

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$ 

This is the formula given in Section 2.2 for the number of elements in the union of two sets. Example 18 illustrates how we can solve counting problems using the subtraction principle.

EXAMPLE 19 A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Solution: To find the number of these applicants who majored neither in computer science nor in business, we can subtract the number of students who majored either in computer science or in business (or both) from the total number of applicants. Let $A_1$ be the set of students who majored in computer science and $A_2$ the set of students who majored in business. Then $A_1 \cup A_2$ is the set of students who majored in computer science or business (or both), and $A_1 \cap A_2$ is the
set of students who majored both in computer science and in business. By the subtraction rule
the number of students who majored either in computer science or in business (or both) equals

\[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316. \]

We conclude that 350 – 316 = 34 of the applicants majored neither in computer science nor in
business.

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20. How many positive integers between 5 and 31
   a) are divisible by 3? Which integers are these?
   Ans: \( 5 \leq 2 \times 3 \) and \( 10 \times 3 \leq 31 \) (note \( 3 \times 11 > 31 \))
   hence \( 10 - 2 + 1 \)

   b) are divisible by 4? Which integers are these?
   Ans: \( 5 \leq 2 \times 4 \) and \( 7 \times 4 \leq 31 \) (note \( 8 \times 4 > 31 \))
   hence \( 7 - 2 + 1 \)

   c) are divisible by 3 and by 4? Which integers are these?
   Ans: First you must compute how many integers are divisible
   by 12. \( 5 \leq 1 \times 12 \) and \( 2 \times 12 \leq 31 \) (note \( 3 \times 12 > 31 \))
   i.e. 12 and 24 are the integers between 5 and 31 that are
divisible by 12.

22. How many positive integers less than 1000
   a) are divisible by 7?
   Ans: \( 7 \leq 1 \times 7 \leq 142 \times 7 \leq 1000 \)
   hence \( 142 - 1 + 1 = 142 \)

   b) are divisible by 7 but not by 11?
   Ans: we are lucky since 7 and 11 are prime numbers, we need to
   compute how many numbers are divisible \( 11 \times 7 = 77 \).
\[
1 \leq 1 * 77 \leq 12 * 77 < 1000
\]
i.e. 12 numbers are divisible by 7 and 11.
Hence: \( 142 - 12 = 130 \)

c) are divisible by both 7 and 11?

Ans: From part (b) \( 12 \)

d) are divisible by either 7 or 11?

Ans: The numbers divisible by 11 is
\[
1 < 1*11 \leq 90*11 < 1000
\]
Let \( A \) = set of numbers divisible by 7
Let \( B \) = set of numbers divisible by 11

\[
| A \cup B | = |A| + |B| - |A \cap B|
\]
\[
| A \cup B | = 142 + 90 - 12 = 220
\]
e) are divisible by exactly one of 7 and 11?

Ans: Part (d) - Part (c) = \( 220 - 12 = 208 \)

f) are divisible by neither 7 nor 11?

Ans: \( 999 - 220 = 779 \)

g) have distinct digits?

Ans: The numbers have three digits and it cannot start with zero.
the numbers are:
\[
\text{FST } \{ \text{F: the first digit; S: Second digit; ..} \}
\]
F cannot start with a zero so it has 9 possibilities
S cannot have the same digits as F
but can include zero: 9 possibilities
T cannot have the same as F and S so 8 possibilities
Total for Three digits: \( 9 * 9 * 8 = 648 \)
Similarly Total for two digits: \( 9 * 9 = 81 \)
Total for one digit = 9
Grand Total: \( 648 + 81 + 9 = 738 \)

h) have distinct digits and are even?

Ans: Single Digits: 2, 4, 6, 8 = 4
Two Digits: The last digit begin 0 is a special case.
\[
10, 20, ..., 90 = 9
\]
The last digit being 2, 4, 6, 8 each has 8 possibilities for the first digit.
(note: ignore, 0 and one from the second digit)
\[
4*8 = 32
\]
Three Digits: The last digit 0 is a special case.
First digit has 9 choices, second has 8.
Hence: \( 9*8 = 72 \)
The last digit being 2, 4, 6, 8. First digit has 8 (no 0
last digit). The second has 8 (ignore first and last)

\[ 4 \times 8 \times 8 = 256 \]

TOTAL: \[ 4 + 9 + 32 + 72 + 256 = 373 \]

*50. How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Ans: First we will calculate how many bits strings are there with 5 consecutive zeros.

All the 0's are in the first five positions from left to right:
There are 5 bits left. (6..10) hence \(2^5 = 32\) strings.

Now 5 zeros in position 2..6. The first bit cannot be a zero, we have counted it before. So the first bit must be 1. Hence there are 4 bits left, hence \(2^4 = 16\)

Now 5 zeros in 3..7 positions. The first bit can take values 0.1 but the second can only be 1.
We have 5 situations like this
Hence \(5 \times 2^4 = 80\)

Total So far: \(80 + 32 = 112\)

By Symmetry we will have 112 strings with 5 consecutive 1's.

But we would have counted 0000011111, 1111100000 twice.

**Hence Total: 112 + 112 - 2 = 222**

6.2 The Pigeonhole Principle
See Examples 5, 6, 7, 8, 9, 10, 11 (we will do the last two together).

**EXAMPLE 10**: During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

**EXAMPLE 11**: Show that among any n + 1 positive integers not exceeding 2n there must be an integer that divides one of the other integers.
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4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

   a) How many balls must she select to be sure of having at least three balls of the same color?

   Ans: Two boxes: Red and Blue
       Let \( n \) be the number of ball requires:
       we need \( \text{Ceil} (n/2) \geq 3 \) \( \text{hence } n = 5. \)

   b) How many balls must she select to be sure of having at least three blue balls?

       She has to take all 10 red balls the 3 more which are blue = 13

#6 Let \( d \) be a positive integer. Show that among any group of \( d + 1 \) (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by \( d \).

Ans: let \( a_0, a_1, a_2, ..., a_d \) be distinct \( d+1 \) integers.

   \[
   a_0 = d \cdot q_0 + r_0 \quad 0 \leq r_0 < d \\
   a_1 = d \cdot q_1 + r_1 \quad 0 \leq r_1 < d \\
   \ldots \\
   a_d = d \cdot q_d + r_d \quad 0 \leq r_d < d \\
   \]

   There are \( 0, 1, 2, ..., (d-1) \) boxes. That is total of \( d \) boxes.
   but there are \( r_0, r_1, ..., r_d \) numbers (\( d+1 \) pigeons) less than \( d \) Hence
   \( \text{Ceil}((d+1)/d) = 2; \)

#16. How many numbers must be selected from the set \{1, 3, 5, 7, 9, 11, 13, 15\} to guarantee that at least one pair of these numbers add up to 16?

Ans: Consider \((1,15), (3,13), (5, 11), (7, 9) \) \( 4 \) pigeon holes
     If you have 5 numbers there will be 2 in the same hole, which adds to 16.
hence answer is 5