Theorem 5: If \( a_n \) is a particular solution for:

\[
 a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)
\]

Then every solution to the above equation is of the form:

\[
 a_n = a_n^h + q_n^h, \quad \text{where } a_n^h \text{ is the solution to the associated homogeneous equation.}
\]

Proof: Let \( a_n^h \) be a particular solution.

\[
 a_n^p = c_1 a_{n-1}^p + c_2 a_{n-2}^p + \ldots + c_k a_{n-k}^p + F(n) \quad (1)
\]

Assume \( b_n \) is the solution for the above equation:

\[
 b_n = c_1 b_{n-1} + c_2 b_{n-2} + \ldots + c_k b_{n-k} + F(n) \quad (2)
\]
\[ b_n - q_n^b = a_n (b_{n-1} - q_{n-1}^b) + c_2 (b_{n-2} - q_{n-2}^b) + \ldots + c_k (b_{n-k} - q_{n-k}^b) \]

\[ x_n = c_1 x_{n-1} + c_2 x_{n-2} + \ldots + c_k x_{n-k} \]

\[ b_n - q_n^h = a_n^h \]

\[ b_n - a_n = a_n^h + a_n \]

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**Sec 8.3**

**Divide-and-Conquer**

\[ f(n) = a \cdot f(\eta^b_n) + g(n) \]

\[ f(\eta^b_n) = a \cdot f(\eta^b_{n+1}) + g(\eta^b_n) \]

\[ f(n) = a \cdot [a \cdot f(\eta^b_n) + g(\eta^b_n)] + g(n) \]

\[ f(n) = a_n \cdot f(\eta^b_n) + a \cdot g(\eta^b_n) + g(n) \]
\[ f(n) = \begin{cases} \log(n) & \text{if } n \leq 1 \\ n^{n/2} & \text{if } n > 1 \end{cases} \]

**Proof:** From the equation, we get

\[ f(n) = \begin{cases} 0 & \text{for } n \leq 0 \\ \log(n) & \text{for } n > 0 \end{cases} \]

Then, \[ f(n) = \begin{cases} \log(n) & \text{for } n > 0 \\ 0 & \text{for } n \leq 0 \end{cases} \]

For \( n > 1 \), let's examine the recursive relation \( f \) is increasing, that each term.

**Theorem:** \( f(n) = \log(n) \) for \( n > 1 \).
\begin{align*}
\frac{1}{2n} &= a \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor - c \\
= f(n) &= a \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor + c \\
\Rightarrow f(n) &= a \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor + c
\end{align*}

\text{Case: } a = 1

\text{Choose } k \text{ such that } \frac{n}{2} = k 

\Rightarrow f(n) = f(k) + c \cdot k

\text{Case: } a = \frac{1}{2}

f(n) = \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor + c \cdot \frac{n}{2}

\Rightarrow f(n) = \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor + c \cdot \frac{n}{2}
\[
\begin{align*}
\log_a n^k &= k \log_a n \\
\iff k &= \frac{\log_a n}{\log_a b} \\
\Rightarrow \quad t &= \frac{\log_b n}{\log_b a} \\
\Rightarrow \quad n &= b^t
\end{align*}
\]

\[\text{where } f(n) \equiv \begin{cases} 
\left( \frac{\log_b a}{\log_b a} \right)^{\log_b n} & a > 1 \\
\left( \log_b n \right) & n = 1
\end{cases}\]